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# Matching of correlators in $\mathrm{AdS}_3/\mathrm{CFT}_2$

**Marika Taylor**

*Institute for Theoretical Physics, University of Amsterdam,*
*Valckenierstraat 65, 1018XE Amsterdam, The Netherlands*
*E-mail:* taylor@science.uva.nl

Abstract: Recently exact agreement has been found between three-point correlators of (single particle) chiral operators computed in string theory on $AdS_3 \times S^3 \times T^4$ with NS-NS flux and those computed in the symmetric orbifold CFT. However, it has also been shown that these correlators disagree with those computed in supergravity, under any identification of single particle operators which respects the symmetries. In this note we resolve this disagreement: the key point is that mixings with multi-particle operators are not suppressed even at large $N$ in extremal correlators. Allowing for such mixings, orbifold/string theory operators and supergravity operators can be matched such that both non-extremal and extremal three point functions agree, giving further evidence for the non-renormalization of the chiral ring.

Keywords: Gauge-gravity correspondence, AdS-CFT Correspondence.

## Contents



## 1. Introduction and summary

Chiral primary operators play an important role in testing the AdS/CFT correspondence. Supersymmetric chiral primaries have protected dimensions, and matching between CFT spectra at weak coupling and supergravity spectra at strong coupling provided the earliest checks of AdS/CFT.

In the correspondence between $\mathcal{N} = 4$ SYM and string theory on $AdS_5 \times S^5$, the three point functions of 1/2 BPS (single trace) operators computed from supergravity were found to match the corresponding correlators computed in free field theory [1]. This indicated the existence of a previously unknown non-renormalization theorem for such correlators which was subsequently proved, modulo various subtleties, in [2]. Moreover, although four and higher point functions of chiral primaries are in general renormalized, there is evidence that extremal correlators, in which the dimension of one operator is equal to the sum of the others, are also protected [3]; see also the review [4].

It is natural to ask whether similar properties for correlators of chiral primaries hold in the case of $\mathrm{AdS}_3/\mathrm{CFT}_2$ dualities. The simplest such case is the D1-D5 system, with $n_1$ D1-branes and $n_5$ D5-branes. Here the duality is between type IIB in an $AdS_3 \times S^3 \times X_4$ background, where $X_4$ is either $T^4$ or $K3$, and a two-dimensional $\mathcal{N} = 4$ superconformal field theory; see for example the review [5].

The bulk and boundary theories in this case are known to have equivalent moduli spaces [6, 7], but they are tractable only at distinct points in the moduli space. In the bulk one can work in the supergravity limit, as one does in the case of $AdS_5 \times S^5$. One can also consider the S-dual system without RR flux, where the string theory is tractable: for Euclidean $AdS_3$ it is described by $H_3^+$ and SU(2) WZW models at level $k = n_5$. The boundary theory becomes tractable in the orbifold limit, namely when the SCFT becomes the symmetric orbifold theory with target space $N = n_1 n_5$ copies of $X_4$. Note that the orbifold theory is not the boundary theory dual to the weakly curved, weakly coupled RR $AdS_3 \times S^3 \times X_4$ background; the boundary theory is a marginal deformation of the orbifold theory, in which the orbifold is resolved.

Whilst the limits in which the boundary and bulk theories are tractable are at different points in the moduli space, matching of the spectrum of chiral primaries is still possible. Comparison of the spectra obtained from supergravity with those of the boundary theory was first carried out in [8, 9]. There were also early attempts to compare three point functions computed from supergravity with those computed in the orbifold CFT. Extremal three point functions were computed in the orbifold CFT in [10, 11] whilst the cubic couplings in supergravity relevant for computing three point functions were determined in [12 − 14]. It was however noted that these cubic couplings do not match in structure the extremal three point functions computed in the orbifold theory. Only the cubic couplings in supergravity, and not the three point functions, were computed in [13, 14]. Computing the three point functions is rather subtle, in that systematic holographic renormalization [15] is required to obtain the correct correlators, satisfying the requisite Ward identities.

Moreover, extremal correlators are subject to additional subtleties: the bulk extremal cubic couplings vanish, and the corresponding three point functions are obtained from finite boundary terms in the action, which in turn should follow from careful reduction of the ten-dimensional action [3]. Put differently, one should first include boundary terms in the ten-dimensional action such that the variational problem is well-posed for the appropriate Dirichlet boundary conditions, and then dimensionally reduce to obtain the effective three-dimensional action.

In practice, it is more convenient to obtain the extremal correlators by analytic continuation of the corresponding non-extremal correlators. That is, one defines the extremal three point functions as

$$\begin{aligned} \langle \mathcal{O}_{\Delta_2+\Delta_3}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\rangle &= \frac{C_{(\Delta_2+\Delta_3)\Delta_2\Delta_3}}{|\vec{x_1}-\vec{x_2}|^{2\Delta_2}|\vec{x_1}-\vec{x_3}|^{2\Delta_3}}; \\ C_{(\Delta_2+\Delta_3)\Delta_2\Delta_3} &= Lim_{\Delta_1\to(\Delta_2+\Delta_3)}(C_{\Delta_1\Delta_2\Delta_3}), \end{aligned} \tag{1.1}$$

where the scalar operator $\mathcal{O}_\Delta$ has dimension $\Delta$, and the non-extremal structure constant $C_{\Delta_1\Delta_2\Delta_3}$ follows from the bulk non-extremal couplings. This analytic continuation was discussed in [3] and more recently such a definition of extremal correlators was discussed in [16] within the framework of holographic renormalization. Note that this approach implicitly assumes that the structure constants are analytic in the operator dimensions, which need not be true, given that the latter are discrete.

Holographically renormalized non-extremal correlators for scalar chiral primaries were recently computed from supergravity in [18], and the corresponding extremal correlators were then determined via analytic continuation. These extremal correlators were compared to those computed in the orbifold CFT. Since only a subset of non-extremal correlators of scalar chiral primaries have so far been computed in the orbifold theory, in [11], only extremal correlators could be compared.

A structural disagreement between these correlators was found. To be more precise, the single particle scalar chiral primaries in the orbifold CFT are labeled by the $(p, p)$



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cohomology of $X_4$, their twist $n \geq 1$, and their R symmetry $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ quantum numbers as

$$\mathcal{O}^{0,0}_{nm\bar{m}}; \qquad \mathcal{O}^{(r)1,1}_{nm\bar{m}}; \qquad \mathcal{O}^{2,2}_{nm\bar{m}}. \tag{1.2}$$

Here $(m, \bar{m})$ are the eigenvalues of $J^3$ and $\bar{J}^3$ respectively and $(r)$ labels the $(1,1)$ cohomology of $X_4$, of dimension $h^{1,1}$; thus $(r)$ runs from 1 to 4 for $T^4$ and from 1 to 20 for $K3$. The operator dimension $\Delta$ is related to the twist and cohomology via

$$\Delta = (n - 1 + p), \tag{1.3}$$

with $J = \bar{J} = \frac{1}{2}\Delta$ being the SO(4) R-symmetry quantum numbers. The cohomology label implicitly defines the transformation properties under the $\mathrm{SO}(h^{1,1})$ global symmetry of the CFT.

On the supergravity side one has a set of operators dual to scalar fields in $AdS_3$ which are labeled by their dimension $\Delta$ and R symmetry $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ quantum numbers:

$$\mathcal{O}^{S^{(a)}}_{\Delta m\bar{m}}; \qquad \mathcal{O}^{\Sigma}_{\Delta m\bar{m}}, \tag{1.4}$$

where $\Delta \geq 1$ for $\mathcal{O}^{S^{(a)}}_{\Delta m\bar{m}}$ and $\Delta \geq 2$ for $\mathcal{O}^{\Sigma}_{\Delta m\bar{m}}$. Here $(a)$ runs from 0 to $h^{1,1}$ and $\Phi \equiv (S^{(a)}, \Sigma)$ are the bulk scalar fields, which couple to these operators. Of the $(h^{1,1}+1)$ operators $\mathcal{O}^{S^{(a)}}_{\Delta m\bar{m}}$ one transforms as a singlet under the $\mathrm{SO}(h^{1,1})$ global symmetry and the remaining $h^{1,1}$ as a vector. $\mathcal{O}^{\Sigma}_{\Delta m\bar{m}}$ is also a singlet under the $\mathrm{SO}(h^{1,1})$ symmetry; see [9] for further details, and tables of the operators.

Already from (1.2) and (1.4) one can see a subtlety in comparing correlators: the identification between orbifold CFT operators and those dual to supergravity fields is not unique, since the protected quantum numbers of dimension, R symmetry charge and the $\mathrm{SO}(h^{1,1})$ global symmetry leave some degeneracy. Given that $\mathcal{O}^{\Sigma}_{1m\bar{m}}$ is absent, a natural identification between orthonormal operators is

$$\begin{aligned} \mathcal{O}^{0,0}_{nm\bar{m}} &\leftrightarrow \mathcal{O}^{S}_{(n-1)m\bar{m}}; \\ \mathcal{O}^{(r)1,1}_{nm\bar{m}} &\leftrightarrow \mathcal{O}^{S^{(r)}}_{nm\bar{m}}; \\ \mathcal{O}^{2,2}_{nm\bar{m}} &\leftrightarrow \mathcal{O}^{\Sigma}_{(n+1)m\bar{m}}, \end{aligned} \tag{1.5}$$

and it is this identification which has been assumed in previous literature, but any linear rotation of this identification such that

$$\begin{pmatrix} \mathcal{O}^{S}_{(n-1)m\bar{m}} \\ \mathcal{O}^{\Sigma}_{(n-1)m\bar{m}} \end{pmatrix} = \mathcal{M} \begin{pmatrix} \mathcal{O}^{0,0}_{nm\bar{m}} \\ \mathcal{O}^{2,2}_{(n-2)m\bar{m}} \end{pmatrix}, \tag{1.6}$$

with $\mathcal{M}$ an arbitrary SO(2) matrix also respects the symmetries and orthonormality.

The result of [18] was that there is a disagreement in the extremal correlators for any choice of $\mathcal{M}$. The disagreement is structural in that for any such linear identification many more of the correlators are non-vanishing in the orbifold theory than in supergravity. Such a disagreement is a priori perhaps not surprising: the computations are at different points in the moduli space and there is no known non-renormalization theorem. Even in



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the $\mathcal{N} = 4$ SYM theory, which has thirty-two supercharges, the non-renormalization of the analogous three point functions is rather subtle and the proof requires assuming the absence of certain conformal invariants [2]. Renormalization in this case, with only sixteen supercharges, is not a priori excluded, particularly as the orbifold theory is a marginal deformation of the actual boundary CFT.

However, recently three point functions of chiral primaries were computed in the NS-NS $AdS_3 \times S^3 \times X_4$ background using the WZW model description of the worldsheet theory. In [23] extremal three point functions of all single particle chiral primaries were computed, whilst in [22] non-extremal three point functions for operators in the $\mathcal{O}^{0,0}_{nm\bar{m}}$ family were computed. Later in [24] the calculations were extended to non-extremal three point functions of all chiral primaries. All of these correlators agree exactly with those computed in the orbifold CFT, although let us recall that in the latter only a subset of the non-extremal correlators have so far been determined.

That computations at different points in the moduli space agree indicates that there is indeed a non-renormalization theorem protecting these correlators, but at the same time raises the puzzle of why the extremal correlators computed from supergravity did not agree with the orbifold CFT (and hence the string) computations. Whilst it is undoubtedly interesting to find that there is a non-renormalization theorem protecting these correlators, it is arguably more important to understand whether there is any unresolved subtle issue in comparing supergravity and dual field theory results. The reason is that in many situations in gravity/gauge theory dualities one wants to use the supergravity description as a tool to compute the strong coupling result, exactly, when no non-renormalization theorem applies.

In this paper we will resolve this issue, and explain how the supergravity correlators are reconciled with the orbifold CFT and string theory correlators. The conclusions are the following. All *non-extremal* three point functions computed via supergravity agree precisely with those computed via string theory provided that the matrix $\mathcal{M}$ is such that

$$\mathcal{M} = \frac{1}{\sqrt{2\Delta}} \begin{pmatrix} (\Delta+1)^{1/2} & -(\Delta-1)^{1/2} \\ (\Delta-1)^{1/2} & (\Delta+1)^{1/2} \end{pmatrix} \tag{1.7}$$

for $\Delta = (n-1) \geq 2$. This agreement provides further evidence for the non-renormalization theorem. Note however that the correspondence between supergravity operators and those in the orbifold CFT is not the naive relation one might have anticipated: $\mathcal{M}$ is not diagonal. This explains the early observation that the cubic couplings in supergravity look very different from the structure constants in the orbifold CFT three point functions.

As discussed in [18] such a linear map between supergravity and orbifold CFT operators is not sufficient to obtain matching for all the extremal correlators. To understand how this issue is resolved, one needs to recall the large $N$ scaling behavior of correlators: the key is that extremal *non-linear* operator mixings are not suppressed in extremal correlators [3]; see also related discussions in Arutyunov:1999en. Thus one can consider an identification between orbifold CFT operators and supergravity operators of the form

$$\mathcal{O}^{p,p}_{\Delta m\bar{m}} \leftrightarrow \mathcal{O}^{\Phi}_{\Delta m\bar{m}} + \frac{1}{\sqrt{N}} \sum_{i,j} b_{ij} \mathcal{O}^{\Phi_i}_{\Delta_i m_i \bar{m}_i} \mathcal{O}^{\Phi_j}_{(\Delta-\Delta_i)(m-m_i)(\bar{m}-\bar{m}_i)} + \cdots, \tag{1.8}$$

where $b_{ij}$ are certain $N$ independent coefficients and the ellipses denote subleading terms in $N$. Such a two particle term contributes at leading order to certain extremal three point functions, but only at subleading order to non-extremal three point functions. Thus with suitable choices of $b_{ij}$ one can match the extremal correlators computed in supergravity with those computed in string theory and the orbifold theory.

The physical interpretation of such non-linear mixings is that single particle string and orbifold CFT operators do not correspond to single particle supergravity operators. At first sight this may seem surprising, since one might have anticipated that the string worldsheet vertex operators for supergravity modes would correspond to single particle supergravity fields, as they do in flat space. However, there is no contradiction: the matching between supergravity fields and string vertex operators is defined by taking the limit of the string computations of $n$-point functions, and comparing with the corresponding supergravity computations. Thus the comparison made here is the correct way to define the relationship between operators dual to supergravity fields and string vertex operators.

So, to summarize, the non-extremal and extremal three point functions computed in supergravity, in string theory and in the orbifold theory agree provided that one correctly matches operators and takes into account certain extremal non-linear operator mixings. Matching of the correlators determines the map between supergravity and orbifold/string theory operators, where quantum numbers alone do not uniquely determine it.

One might wish to explore whether other correlators are protected by non-renormalization theorems. In the analogous case of $\mathcal{N} = 4$ SYM, there is evidence that extremal (and next to extremal) $n$-point functions of chiral primaries are similarly protected, see [4], and thus it is possible that all extremal correlators in this case too will match between string theory, the orbifold CFT and supergravity. Note however that general non-extremal $n$-point functions for $n \geq 4$ are not protected even in $\mathcal{N} = 4$ SYM, and are thus unlikely to be protected in this less supersymmetric system. Comparison of the extremal correlators will again be subtle since non-linear operator mixings may contribute at leading order. That is, in an extremal $n$-point function mixings of the type

$$\mathcal{O}^{\Phi}_{\Delta} + \frac{1}{N^{(n-2)/2}} \prod_{i=1}^{n-1} \mathcal{O}^{\Phi_i}_{\Delta_i} + \cdots \tag{1.9}$$

with $\sum_i \Delta_i = \Delta$ are not suppressed.

An important open issue is to understand better when there are non-renormalization theorems for correlators. The (almost) proof of the non-renormalization of three point functions in $\mathcal{N} = 4$ SYM relies on sophisticated harmonic superspace techniques. In this case one should be able to use the $2d$ $\mathcal{N} = 4$ supersymmetry to explain the non-renormalization. However, an understanding of the non-renormalization from the bulk supergravity perspective would more immediately generalize to other AdS/CFT dualities. Such a non-renormalization theorem in the bulk would involve arguing that $\alpha'$ corrections to the onshell renormalized supergravity action do not contribute to the correlators.

The above discussion relates to three point functions of chiral primaries associated with single particle supergravity fields. Given that these appear to satisfy a non-renormalization theorem, with appropriate operator mixing taken into account, it seems very likely that all three point functions of multi-particle chiral primaries are similarly protected. In the analogous case of $\mathcal{N} = 4$ SYM, there is indeed evidence for this, from both harmonic superspace considerations [2] and more recently from the holographic analysis of LLM bubbling solutions in [20].

If indeed all three point functions are protected, then an immediate consequence would be that vevs of chiral primary operators in states created by other chiral primaries are also not renormalized. Now in [17 − 19] such vevs were used to test the proposed correspondence between 1/2 BPS D1-D5 fuzzball geometries and superpositions of RR ground states. Non-renormalization of the vevs can be used to push this correspondence much further, as will be explored in a separate publication.

The plan of this paper is as follows. In section 2 we review the results of the supergravity computation of three point functions, whilst in section 3 the results of the corresponding string theory computations are reviewed. In section 4 the non-extremal correlators are found to match, with an appropriate linear identification of operators. In section 5 it is shown that additional non-linear terms in this operator identification are needed to obtain matching of extremal correlators.

## 2. Supergravity computation of correlators

In this section we will review the holographic computation of three point functions of single particle scalar chiral primaries. These correlators are computed by perturbing about the $AdS_3 \times S^3 \times X_4$ background, where $X_4$ is $T^4$ or $K3$. There are three distinct families of scalar chiral primaries associated with the $(p, p)$ cohomology of $X_4$ with $p = 0, 1, 2$ respectively. The operators couple to the following scalar fields in $AdS_3$:

$$S_{kI}; \qquad S^{(r)}_{kI}; \qquad \Sigma_{kI}. \tag{2.1}$$

Here $(k, I)$ denote the SO(4) R symmetry labels: $k$ is the degree of the associated $S^3$ scalar spherical harmonic and $I$ denotes the remaining Dynkin labels. Expressing $\mathrm{SO}(4) = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$, appropriate labels are the $J^3_{L/R}$ charges $(m, \bar{m})$. The middle cohomology of $X_4$ is labeled by $(r) = 1, \cdots h^{1,1}(X_4)$ where $h^{1,1}$ is four for $T^4$ and twenty for $K3$; this label defines the field transformations under the global $\mathrm{SO}(h^{1,1})$ symmetry.

Up to the overall normalization factor, the kinetic terms for these fields are canonically normalized, namely the bulk action is

$$\begin{aligned}\mathcal{S} = \frac{N}{4\pi} \int d^3x \sqrt{-G} \Big( & R_G + 2 - \frac{1}{2}((DS_{kI})^2 - k(k-2)(S_{kI})^2) \\ & - \frac{1}{2}((DS^{(r)}_{kI})^2 - k(k-2)(S^{(r)}_{kI})^2) - \frac{1}{2}((D\Sigma_{kI})^2 - k(k-2)\Sigma^2_{kI}) + \cdots \Big).\end{aligned} \tag{2.2}$$

The overall normalization is proportional to the integer $N = n_1 n_5$. Note that the mass terms are such that the scalar fields associated with degree $k$ harmonics couple to operators of dimension $k$. For the $S$ fields $k \geq 1$ whilst for the $\Sigma$ fields $k \geq 2$. To calculate the three point functions one also needs the appropriate cubic couplings computed in [12, 13]. These are given by

$$
\begin{aligned}
-\frac{N}{4\pi}\int d^3x\sqrt{-G}(T_{123}S^{(a)1}S^{(a)2}\Sigma^3 + U_{123}\Sigma^1\Sigma^2\Sigma^3) \quad (2.3)\\
\equiv -\frac{N}{16\pi}\int d^3x\sqrt{-G}V_{123}\times \\
\times\left(\frac{S^{(a)1}S^{(a)2}\Sigma^3}{\sqrt{(k_1+1)(k_2+1)}} + \frac{(k_1^2+k_2^2+k_3^2-2)}{(k_1+1)(k_2+1)}\frac{\Sigma^1\Sigma^2\Sigma^3}{6\sqrt{(k_1-1)(k_2-1)}}\right),\\
V_{123} = \frac{\Sigma(\Sigma+2)(\Sigma-2)\alpha_1\alpha_2\alpha_3 a_{123}}{(k_3+1)\sqrt{k_1k_2k_3(k_3-1)}}
\end{aligned}
$$

where $k_i$ denotes the dimension of the operator dual to the field $\Psi^i$, $\Sigma = k_1+k_2+k_3$, $\alpha_1 = \frac{1}{2}(k_2+k_3-k_1)$ etc and $a_{123}$ is shorthand for the spherical harmonic overlap. Here the label $(a) = 1,\cdots h^{1,1}(X_4)+1 \equiv n$ includes all $S$ fields. For subsequent notational convenience we introduce the combinations $T_{123}$ and $U_{123}$ which are defined implicitly by the above equalities. Compactification of type IIB on $X_4$ gives rise to a theory with SO$(n)$ symmetry, and the cubic couplings respect this symmetry. Note however that is an accidental symmetry: only the SO$(h^{(1,1)})$ symmetry is respected by the orbifold CFT and string theory three point functions.

The (renormalized) correlators can be computed using standard holographic renormalization techniques. The two point functions are [18]:

$$
\begin{aligned}
\langle\mathcal{O}^{S^{(a)}}_{k_1I_1}(x)\mathcal{O}^{S^{(b)}}_{k_2I_2}(0)\rangle_h &= \frac{N}{2\pi^2}(k_1-1)^2\left(\frac{1}{x^{2k_1}}\right)_R\delta_{I_1I_2}\delta_{k_1k_2}\delta^{(a)(b)}; \qquad k\neq 1 \quad (2.4)\\
\langle\mathcal{O}^{\Sigma}_{k_1I_1}(x)\mathcal{O}^{\Sigma}_{k_2I_2}(0)\rangle_h &= \frac{N}{2\pi^2}(k_1-1)^2\left(\frac{1}{x^{2k_1}}\right)_R\delta_{I_1I_2}\delta_{k_1k_2},
\end{aligned}
$$

where $\mathcal{O}^{S^{(a)}}$ and $\mathcal{O}^{\Sigma}$ denote the operators dual to $S^{(a)}$ and $\Sigma$ respectively. The subscript $R$ indicates that the expressions are renormalized whilst the subscript $h$ in these and subsequent expressions denotes that these are the holographically computed correlators. When $k=1$, $(k-1)$ is replaced by 1 in the first expression; this is a special case in which the Breitenlohner-Freedman bound is saturated. Recall that there is no $k=1$ operator $\mathcal{O}_{\Sigma}$.

The three point functions are [18]:

$$
\begin{aligned}
\langle\mathcal{O}^{S^{(a)}}(x_1)\mathcal{O}^{S^{(b)}}(x_2)\mathcal{O}^{\Sigma}(x_3)\rangle_h &= \frac{N}{4\pi^3}\frac{W_{123}T_{123}\delta^{(a)(b)}}{|\vec{x}_1-\vec{x}_2|^{2\alpha_3}|\vec{x}_1-\vec{x}_3|^{2\alpha_2}|\vec{x}_2-\vec{x}_3|^{2\alpha_1}}; \quad (2.5)\\
\langle\mathcal{O}^{\Sigma}(x_1)\mathcal{O}^{\Sigma}(x_2)\mathcal{O}^{\Sigma}(x_3)\rangle_h &= \frac{3N}{4\pi^3}\frac{W_{123}U_{123}}{|\vec{x}_1-\vec{x}_2|^{2\alpha_3}|\vec{x}_1-\vec{x}_3|^{2\alpha_2}|\vec{x}_2-\vec{x}_3|^{2\alpha_1}};\\
W_{123} &= \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\frac{1}{2}(\Sigma-2))}{\Gamma(k_1-1)\Gamma(k_2-1)\Gamma(k_3-1)}.
\end{aligned}
$$

Here the operator at position $x_i$ has dimension $k_i$ and SO(4) R symmetry labels $I_i$.

To compare with the orbifold CFT and string theory computations one wants normalized three point functions, dividing out by the norms of the operators as given by the two point functions. Suppressing the standard position dependence, this gives:

$$\begin{aligned}
\langle \hat{\mathcal{O}}^{S^{(a)}} \hat{\mathcal{O}}^{S^{(b)}} \hat{\mathcal{O}}^{\Sigma} \rangle_h &= \frac{1}{\sqrt{2N}} \tilde{W}_{123} T_{123} \delta^{(a)(b)}; \\
\langle \hat{\mathcal{O}}^{\Sigma} \hat{\mathcal{O}}^{\Sigma} \hat{\mathcal{O}}^{\Sigma} \rangle_h &= \frac{3}{\sqrt{2N}} \tilde{W}_{123} U_{123}; \\
\tilde{W}_{123} &= \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\frac{1}{2}(\Sigma - 2))}{\Gamma(k_1)\Gamma(k_2)\Gamma(k_3)},
\end{aligned} \tag{2.6}$$

where $\hat{O}$ denotes the unit normalized operators. The remaining correlators vanish

$$\langle \hat{\mathcal{O}}^{S^{(a)}} \hat{\mathcal{O}}^{S^{(b)}} \hat{\mathcal{O}}^{S^{(c)}} \rangle_h = \langle \hat{\mathcal{O}}^{S^{(a)}} \hat{\mathcal{O}}^{\Sigma} \hat{\mathcal{O}}^{\Sigma} \rangle_h = 0, \tag{2.7}$$

regardless of the operator dimension.

For later purposes it will be useful to give explicitly extremal correlators in which one of the $\alpha_i = 0$. These are *defined* as the continuation of the expressions (2.6): the pole in $\tilde{W}_{123}$ as one of the $\alpha_i \to 0$ cancels a corresponding zero in the bulk couplings $(T_{123}, U_{123})$ to give a finite limit. The relevant normalized extremal three point functions are thus of three types:

$$\begin{aligned}
\langle \hat{\mathcal{O}}^{S^{(a)}\dagger}_{k_1+k_2} \hat{\mathcal{O}}^{S^{(b)}}_{k_1} \hat{\mathcal{O}}^{\Sigma}_{k_2} \rangle_h &= \delta^{(a)(b)} \frac{a_{123}}{\sqrt{N}} \sqrt{\frac{2k_1k_2(k_1+k_2)(k_1+k_2+1)}{(k_1+1)(k_2+1)^2(k_2-1)}}; \\
\langle \hat{\mathcal{O}}^{\Sigma\dagger}_{k_1+k_2} \hat{\mathcal{O}}^{S^{(a)}}_{k_1} \hat{\mathcal{O}}^{S^{(b)}}_{k_2} \rangle_h &= \delta^{(a)(b)} \frac{a_{123}}{\sqrt{N}} \sqrt{\frac{2k_1k_2(k_1+k_2)}{(k_1+1)(k_2+1)(k_1+k_2-1)}}; \\
\langle \hat{\mathcal{O}}^{\Sigma\dagger}_{k_1+k_2} \hat{\mathcal{O}}^{\Sigma}_{k_1} \hat{\mathcal{O}}^{\Sigma}_{k_2} \rangle_h &= \frac{a_{123}}{\sqrt{N}} \frac{(k_1^2+k_2^2+(k_1+k_2)^2-2)^2}{(k_1+1)(k_2+1)} \sqrt{\frac{k_1k_2(k_1+k_2)}{2(k_1-1)(k_2-1)(k_1+k_2-1)}}.
\end{aligned} \tag{2.8}$$

Note that the triple overlap $a_{123} = 1$ when the operator with maximum dimension also has SO(2) R charges which are minus the sums of the SO(2) charges of the other operators. In particular that the extremal correlators at lowest dimension are:

$$\langle \hat{\mathcal{O}}^{\Sigma\dagger}_{2} \hat{\mathcal{O}}^{S^{(a)}}_{1} \hat{\mathcal{O}}^{S^{(b)}}_{1} \rangle_h = \delta^{(a)(b)} \frac{a_{123}}{\sqrt{N}}; \qquad \langle \hat{\mathcal{O}}^{S^{(c)}\dagger}_{2} \hat{\mathcal{O}}^{S^{(a)}}_{1} \hat{\mathcal{O}}^{S^{(b)}}_{1} \rangle_h = 0. \tag{2.9}$$

## 3. String theory/orbifold CFT correlators

In this section we will review the results for the corresponding correlators computed in string theory and the orbifold CFT. Not all non-extremal correlators have been computed in the orbifold CFT, but those which are known agree with those computed via the string theory, as do all extremal correlators. Here we summarize the results of [22 − 24] for the string theory computation of three point functions of scalar chiral primaries. Note that general three point functions involving vector chiral primaries associated with the (0, 2) and (2, 0) cohomology of $X_4$ are also given in [23, 24]. We will not consider these here,



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since the corresponding holographic correlators have not been computed, but it should be straightforward to extend our discussions to these operators.

The scalar chiral primaries are labeled by the $(p,p)$ cohomology of $X_4$, their twist $n \geq 1$, and their R symmetry $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ quantum numbers as

$$\mathcal{O}^{0,0}_{nm\bar{m}}; \qquad \mathcal{O}^{(r)1,1}_{nm\bar{m}}; \qquad \mathcal{O}^{2,2}_{nm\bar{m}}. \tag{3.1}$$

Here $(m,\bar{m})$ are the eigenvalues of $J^3$ and $\bar{J}^3$ respectively and $(r)$ labels the $(1,1)$ cohomology of $X_4$, of dimension $h^{1,1}$; thus $(r)$ runs from 1 to 4 for $T^4$ and from 1 to 20 for $K3$. The cohomology label is equivalent to giving the transformation properties under the $\mathrm{SO}(h^{1,1})$ global symmetry. The operator dimension is given by

$$\Delta = (n - 1 + p), \tag{3.2}$$

with $J = \bar{J} = \frac{1}{2}\Delta$. These operators are orthonormal

$$\langle \mathcal{O}^{p,p}_{n-m-\bar{m}} \mathcal{O}^{p,p}_{nm\bar{m}} \rangle_s = 1, \tag{3.3}$$

with the subscript $s$ denoting that these are string theory correlators. The three point functions can be conveniently expressed as

$$\begin{aligned}
\langle \mathcal{O}^{\epsilon_1,\bar{\epsilon}_1}_{n_1m_1\bar{m}_1} \mathcal{O}^{\epsilon_2,\bar{\epsilon}_2}_{n_2m_2\bar{m}_2} \mathcal{O}^{\epsilon_3,\bar{\epsilon}_3}_{n_3m_3\bar{m}_3} \rangle_s &= \frac{1}{\sqrt{N}} L(J_i, m_i) L(\bar{J}_i, \bar{m}_i) \frac{(\sum_{i=1}^3 \epsilon_i n_i + 1)(\sum_{i=1}^3 \bar{\epsilon}_i n_i + 1)}{4(n_1n_2n_3)^{1/2}}; \\
\langle \mathcal{O}^{(r)1,1}_{n_1m_1\bar{m}_1} \mathcal{O}^{(s)1,1}_{n_2m_2\bar{m}_2} \mathcal{O}^{\epsilon,\bar{\epsilon}}_{n_3m_3\bar{m}_3} \rangle_s &= \frac{1}{\sqrt{N}} \delta^{(r)(s)} L(J_i, m_i) L(\bar{J}_i, , \bar{m}_i) \left( \frac{n_1 n_2}{n_3} \right)^{1/2};
\end{aligned} \tag{3.4}$$

where $\epsilon = (p-1)$ for $p = 0, 2$. Here

$$\begin{aligned}
L(J_i, m_i) &= d^{J_1,J_2,J_3}_{m_1,m_2,m_3} \eta_{J_i} \left( \frac{\alpha_1! \alpha_2! \alpha_3! (J_1 + J_2 + J_3 + 1)!}{(2J_1)!(2J_2)!(2J_3)!} \right)^{1/2}, \\
\eta_{J_i} &= (-)^{\frac{1}{2}(J_1+J_2+J_3)}.
\end{aligned} \tag{3.5}$$

with $\alpha_1 = J_2 + J_3 - J_1 = \frac{1}{2}(\Delta_2 + \Delta_3 - \Delta_1)$ etc and

$$d^{J_1,J_2,J_3}_{m_1,m_2,m_3} = \begin{pmatrix} J_1 & J_2 & J_3 \\ m_1 & m_2 & m_3 \end{pmatrix}, \tag{3.6}$$

are the SU(2) $3j$ symbols. Note that U(1) R-charge conservation enforces that $m_1 + m_2 + m_3 = \bar{m}_1 + \bar{m}_2 + \bar{m}_3 = 0$ in the correlators.

## 4. Matching non-extremal correlators

Let us now consider the matching of the correlators (2.6), (2.7) and (3.4). The most general *linear* identification of operators which respects the symmetries is

$$\begin{aligned}
\mathbb{O}^{S^{(r)}}_{nm\bar{m}} &\leftrightarrow \mathcal{O}^{(r)1,1}_{nm\bar{m}}; \\
\begin{pmatrix} \mathbb{O}^{S}_{(n-1)m\bar{m}} \\ \mathbb{O}^{\Sigma}_{(n-1)m\bar{m}} \end{pmatrix} &= \mathcal{M} \begin{pmatrix} \mathcal{O}^{0,0}_{nm\bar{m}} \\ \mathcal{O}^{2,2}_{(n-2)m\bar{m}} \end{pmatrix},
\end{aligned} \tag{4.1}$$

for an arbitrary SO(2) matrix $\mathcal{M}$. Here we denote by $\mathbb{O}^{\Phi}_{\Delta}$ operators which for *non-extremal* correlators are to be identified with the holographic operators $\hat{\mathcal{O}}^{\Phi}_{\Delta}$. For *extremal* correlators we will need to refine the map between these operators and the holographic operators.

The matrix $\mathcal{M}$ is completely fixed by the vanishing of the correlators

$$\langle \mathbb{O}^{S} \mathbb{O}^{S^{(r)}} \mathbb{O}^{S^{(s)}} \rangle = 0, \tag{4.2}$$

which implies

$$\mathcal{M} = \frac{1}{\sqrt{2\Delta}} \begin{pmatrix} (\Delta+1)^{1/2} & -(\Delta-1)^{1/2} \\ (\Delta-1)^{1/2} & (\Delta+1)^{1/2} \end{pmatrix} \tag{4.3}$$

for $\Delta = (n-1) \geq 2$. Clearly there can be no operator mixing at $\Delta = 1$, since there are no dimension one $\mathbb{O}^{\Sigma}$ operators; this is hence a special case which will be discussed separately.

Having determined $\mathcal{M}$, there is no further freedom in the operator identification and one can check whether the remaining correlators agree. Forming the appropriate linear combinations of the string theory correlators (3.4), one finds that

$$\begin{aligned} \langle \mathbb{O}^{S^{(a)}} \mathbb{O}^{S^{(b)}} \mathbb{O}^{S^{(c)}} \rangle_s &= 0; \qquad \Delta_i \neq 1, \\ \langle \mathbb{O}^{S^{(a)}} \mathbb{O}^{\Sigma} \mathbb{O}^{\Sigma} \rangle_s &= 0; \qquad \Delta_1 \neq 1, \end{aligned} \tag{4.4}$$

$$\langle \mathbb{O}^{\Sigma} \mathbb{O}^{S^{(a)}} \mathbb{O}^{S^{(b)}} \rangle_s = \frac{1}{\sqrt{N}} L(J_i, m_i) L(\bar{J}_i, \bar{m}_i) \delta^{(a)(b)} \frac{\sqrt{2}(\Delta_1\Delta_2\Delta_3)^{1/2}}{(\Delta_1^2-1)^{1/2}}; \tag{4.5}$$

$$\langle \mathbb{O}^{\Sigma} \mathbb{O}^{\Sigma} \mathbb{O}^{\Sigma} \rangle_s = \frac{(\Delta_1^2+\Delta_2^2+\Delta_3^2-2)}{2((\Delta_2^2-1)(\Delta_3^2-1))^{1/2}} \langle \mathbb{O}^{\Sigma} \mathbb{O}^{S^{(a)}} \mathbb{O}^{S^{(b)}} \rangle_s. \tag{4.6}$$

Here the subscript $s$ denotes that these are linear combinations of the correlators computed in the string theory.

Now let us compare these correlators with the holographic correlators (2.6) and (2.7): the zeroes given in (2.7) are reproduced (except in the special cases involving dimension one operators). Moreover, using (2.6) and noting that

$$\frac{3U_{123}}{T_{231}} = \frac{(\Delta_1^2+\Delta_2^2+\Delta_3^2-2)}{2((\Delta_2^2-1)(\Delta_3^2-1))^{1/2}}, \tag{4.7}$$

one sees that the ratio given in (4.6) indeed agrees with that from supergravity. Thus one need only compare the overall normalization of (4.5) with that of the supergravity correlator for all non-extremal correlators to match. Noting that

$$\tilde{W}_{123} T_{231} = \frac{\alpha_1!\alpha_2!\alpha_3!(J_1+J_2+J_3+1)!}{(2J_1)!(2J_2)!(2J_3)!} \frac{\sqrt{\Delta_1\Delta_2\Delta_3}}{(\Delta_1^2-1)^{1/2}} \frac{2a_{123}}{((\Delta_1+1)(\Delta_2+1)(\Delta_3+1))^{1/2}}, \tag{4.8}$$

the holographic correlator (2.6) can be rewritten in terms of the string theory correlator as

$$\frac{\langle \hat{\mathcal{O}}^{\Sigma} \hat{\mathcal{O}}^{S^{(a)}} \hat{\mathcal{O}}^{S^{(b)}} \rangle_h}{\langle \mathbb{O}^{\Sigma} \mathbb{O}^{S^{(a)}} \mathbb{O}^{S^{(b)}} \rangle_s} = \frac{1}{\eta_{J_i}\eta_{\bar{J}_i} d^{J_1,J_2J_3}_{m_1,m_2,m_3} d^{\bar{J}_1,\bar{J}_2\bar{J}_3}_{\bar{m}_1,\bar{m}_2,\bar{m}_3}} \frac{a_{123}}{((\Delta_1+1)(\Delta_2+1)(\Delta_3+1))^{1/2}}. \tag{4.9}$$

Triple integrals of spherical harmonics can be expressed in terms of $3j$ symbols; in particular the triple overlap $a_{123}$ can be written as [26]

$$a_{123} = \eta_{J_i}\eta_{\bar{J}_i} d^{J_1,J_2J_3}_{m_1,m_2,m_3} d^{\bar{J}_1,\bar{J}_2\bar{J}_3}_{\bar{m}_1,\bar{m}_2,\bar{m}_3} ((2J_1+1)(2J_2+1)(2J_3+1))^{1/2}, \tag{4.10}$$



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and thus the normalization of the holographic correlators precisely matches that of the string correlators!

## 5. Matching of exceptional extremal correlators

The linear matching between supergravity and orbifold CFT operators is sufficient for all non-extremal correlators, and most extremal correlators, to match. There remains however a discrepancy for correlators involving dimension one operators, where no linear mixing was possible.

For correlators which involve at least one dimension one operator, some agree with the holographic results, namely

$$\begin{aligned} \langle \mathbb{O}^S_{\Delta_2} \mathbb{O}^{S(r)}_1 \mathbb{O}^{S(s)}_{\Delta_3} \rangle_s &= 0; \\ \langle \mathbb{O}^\Sigma_{\Delta_2} \mathbb{O}^{S(r)}_1 \mathbb{O}^{S(s)}_{\Delta_3} \rangle_s &= \frac{1}{\sqrt{N}} L(J_i, m_i) L(\bar{J}_i, \bar{m}_i) \delta^{(r)(s)} \frac{\sqrt{2\Delta_2\Delta_3}}{(\Delta_2^2 - 1)^{1/2}}; \\ \langle \mathbb{O}^S_1 \mathbb{O}^S_{\Delta_2} \mathbb{O}^\Sigma_{\Delta_3} \rangle_s &= \frac{1}{\sqrt{N}} L(J_i, m_i) L(\bar{J}_i, \bar{m}_i) \frac{\sqrt{2\Delta_2\Delta_3}}{(\Delta_3^2 - 1)^{1/2}}, \end{aligned} \tag{5.1}$$

but the rest do not:

$$\begin{aligned} \langle \mathbb{O}^S_1 \mathbb{O}^S_1 \mathbb{O}^S_2 \rangle_s &= \frac{2}{\sqrt{N}} L(J_i, m_i) L(\bar{J}_i, \bar{m}_i); \\ \langle \mathbb{O}^S_1 \mathbb{O}^S_1 \mathbb{O}^\Sigma_2 \rangle_s &= \frac{\sqrt{3}}{\sqrt{N}} L(J_i, m_i) L(\bar{J}_i, \bar{m}_i); \\ \langle \mathbb{O}^S_1 \mathbb{O}^{S(r)}_{\Delta_2} \mathbb{O}^{S(s)}_{\Delta_3} \rangle_s &= \frac{1}{\sqrt{N}} L(J_i, m_i) L(\bar{J}_i, \bar{m}_i) \left( \frac{\Delta_2\Delta_3}{2} \right)^{1/2}; \\ \langle \mathbb{O}^S_1 \mathbb{O}^S_{\Delta_2} \mathbb{O}^S_{\Delta_3} \rangle_s &= \frac{1}{\sqrt{2N}} L(J_i, m_i) L(\bar{J}_i, \bar{m}_i) \sqrt{\Delta_2\Delta_3}; \\ \langle \mathbb{O}^S_1 \mathbb{O}^\Sigma_{\Delta_2} \mathbb{O}^\Sigma_{\Delta_3} \rangle_s &= \frac{1}{2\sqrt{2N}} L(J_i, m_i) L(\bar{J}_i, \bar{m}_i) \frac{\sqrt{\Delta_2\Delta_3}}{((\Delta_2^2 - 1)(\Delta_3^2 - 1))^{1/2}} (\Delta_2^2 + \Delta_3^2 - 1), \end{aligned} \tag{5.2}$$

with the corresponding holographic correlators being

$$\begin{aligned} \langle \hat{\mathcal{O}}^S_1 \hat{\mathcal{O}}^S_{\Delta_2} \hat{\mathcal{O}}^S_{\Delta_3} \rangle_h &= \langle \hat{\mathcal{O}}^S_1 \hat{\mathcal{O}}^{S(r)}_{\Delta_2} \hat{\mathcal{O}}^{S(s)}_{\Delta_3} \rangle_h = \langle \hat{\mathcal{O}}^S_1 \hat{\mathcal{O}}^\Sigma_{\Delta_2} \hat{\mathcal{O}}^\Sigma_{\Delta_3} \rangle_h = 0; \\ \langle \hat{\mathcal{O}}^S_1 \hat{\mathcal{O}}^S_1 \hat{\mathcal{O}}^\Sigma_2 \rangle_h &= \frac{1}{\sqrt{N}} a_{123} \equiv \frac{2}{\sqrt{3N}} L(J_i, m_i) L(\bar{J}_i, \bar{m}_i). \end{aligned} \tag{5.3}$$

Note that all these correlators are extremal because the spherical harmonic triple overlaps are only non-zero when $\Delta_3 = (\Delta_2 \pm 1)$. This follows from the addition of SO(4) representations

$$\left( \frac{1}{2}, \frac{1}{2} \right) \oplus \left( \frac{\Delta_2}{2}, \frac{\Delta_2}{2} \right) \rightarrow \left( \frac{\Delta_2 \pm 1}{2}, \frac{\Delta_2 \pm 1}{2} \right). \tag{5.4}$$

Whilst the extremal holographic correlators are (by construction) the analytic continuation of corresponding non-extremal correlators, the string theory correlators in (5.2) are not the analytic continuation of corresponding non-extremal correlators given in (4.5). As we will now explain, this apparent discrepancy between extremal correlators can be resolved by allowing for *non-linear* operator mixing.



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### 5.1 Large $N$ behavior of correlators

Let $\mathcal{O}_k^{\Phi}$ denote the operator of dimension $k$ and SO(2) R charges $(k/2, k/2)$ dual to the supergravity field $\Phi$, where $\Phi = (S^{(a)}, \Sigma)$. Now denote by

$$\mathcal{O}_k^{[\Phi]_n} = \left[ \prod_{i=1}^{n} \mathcal{O}_{k_i}^{\Phi_i} \right] \tag{5.5}$$

the associated protected $n$-particle operators, with dimension $k = \sum_i k_i$ and R charges $(k/2, k/2)$. Here $[\mathcal{O}_{k_i}^{\Phi_i} \cdots]$ denotes the highest weight component of the direct product of SO(4) representations.

The operators $\mathcal{O}_k^{[\Phi]_n}$ transform in the same SO(4) representation as the single particle operators $\mathcal{O}_k^{\Phi}$ and therefore one would anticipate that there is operator mixing. Although generically operator mixing with multi-particle operators is suppressed in the large $N$ limit, this is not true for operators transforming in the same representations. One can understand this from large $N$ counting arguments as follows.

Consider first correlation functions of single particle operators. The two and three point functions computed from gravity scale as $N$, as given in (2.4) and (2.5), so the normalized two and three point functions scale as one and $1/\sqrt{N}$ respectively. Four point functions include both disconnected and connected contributions. The former scale as $N^2$ and are such that

$$\langle \mathcal{O}_{k_1}^{\Phi_1}(x_1) \mathcal{O}_{k_2}^{\Phi_2}(x_2) \mathcal{O}_{k_3}^{\Phi_3}(x_3) \mathcal{O}_{k_4}^{\Phi_4}(x_4) \rangle = N^2 \left( \delta^{\Phi_1 \Phi_2} \delta^{\Phi_3 \Phi_4} \frac{\delta(k_1 + k_2) \delta(k_3 + k_4)}{x_{12}^{2|k_1|} x_{34}^{2|k_3|}} + \cdots \right), \tag{5.6}$$

where the ellipses denote permutations and numerical factors are suppressed. The scaling as $N^2$ follows from the fact that these disconnected contributions are the products of two point functions. Working with unit normalized operators, the disconnected contribution to the four point function thus scales as one. Connected contributions to four point functions however scale as $N$ or, working with unit normalized operators, as $1/N$. Note that holographic computation of the connected contributions involves both the cubic and quartic couplings, whilst the disconnected contributions follow entirely from the (renormalized) quadratic action.

Now let us consider correlation functions involving multi particle operators. In particular, one can read off the large $N$ behavior of correlators involving double particle operators from the single particle correlators discussed above. The operator product expansion $\mathcal{O}_{k_1}^{\Phi_1}(x_1) \mathcal{O}_{k_2}^{\Phi_2}(x_2)$ contains the term

$$\mathcal{O}_{k_1}^{\Phi_1}(x_1) \mathcal{O}_{k_2}^{\Phi_2}(x_2) \rightarrow [\mathcal{O}_{k_1}^{\Phi_1}(x_1) \mathcal{O}_{k_2}^{\Phi_2}(x_1)] \tag{5.7}$$

with unit coefficient as $x_1 \rightarrow x_2$ since the double particle operator is defined by the short distance limit. Thus from the $x_1 \rightarrow x_2$ behavior of correlators one can extract the $N$ scaling of mixed correlators involving both single and multi particle operators. Working with unit normalized single particle operators this gives

$$
\begin{aligned}
\langle \hat{\mathcal{O}}_k^{\dagger\Phi_1}(x_1)[\hat{\mathcal{O}}_{k_1}^{\Phi_2}(x_2)\hat{\mathcal{O}}_{k_2}^{\Phi_3}(x_2)]\rangle &\approx \frac{C^{123}_{k_1k_2}}{\sqrt{N}}; \qquad (5.8)\\
\langle [\hat{\mathcal{O}}_l^{\Phi_1}(x_1)\hat{\mathcal{O}}_{k-l}^{\Phi_2}(x_1)]^\dagger \hat{\mathcal{O}}_{k_1}^{\Phi_3}(x_2)\hat{\mathcal{O}}_{k_2}^{\Phi_4}(x_3)\rangle &\approx (\delta^{lk_1}\delta^{\Phi_1\Phi_3}\delta^{\Phi_2\Phi_4} + \delta^{lk_2}\delta^{\Phi_1\Phi_4}\delta^{\Phi_2\Phi_3});\\
\langle [\hat{\mathcal{O}}_l^{\Phi_1}(x_1)\hat{\mathcal{O}}_{k-l}^{\Phi_2}(x_1)]^\dagger \hat{\mathcal{O}}_{k_1}^{\Phi_3}(x_2)\hat{\mathcal{O}}_{k_2}^{\Phi_4}(x_3)\rangle &\approx \frac{1}{N}; \qquad l \neq k_1, k_2;\\
\langle [\hat{\mathcal{O}}_l^{\Phi_1}(x_1)\hat{\mathcal{O}}_{k-l}^{\Phi_2}(x_1)]^\dagger [\hat{\mathcal{O}}_{k_1}^{\Phi_3}(x_2)\hat{\mathcal{O}}_{k_2}^{\Phi_4}(x_2)]\rangle &\approx (\delta^{lk_1}\delta^{\Phi_1\Phi_3}\delta^{\Phi_2\Phi_4} + \delta^{lk_2}\delta^{\Phi_1\Phi_4}\delta^{\Phi_2\Phi_3});\\
\langle [\hat{\mathcal{O}}_l^{\Phi_1}(x_1)\hat{\mathcal{O}}_{k-l}^{\Phi_2}(x_1)]^\dagger [\hat{\mathcal{O}}_{k_1}^{\Phi_3}(x_2)\hat{\mathcal{O}}_{k_2}^{\Phi_4}(x_2)]\rangle &\approx \frac{1}{N}; \qquad l \neq k_1, k_2.
\end{aligned}
$$

where in all cases $k = k_1 + k_2$, and thus the correlators are extremal. The (standard) $x_i$ dependence of the correlators is suppressed. Here structure constants $C^{123}_{k_1k_2}$ follow from the extremal single particle correlators given in (2.8). The second and fourth correlators follow from the disconnected components of the four point functions, whilst the third and fifth correlators pick up contributions only from the connected components and are thus subleading.

This large $N$ counting demonstrates that operator mixings which are extremal are not suppressed in extremal correlators. That is, suppose one considers operators such that

$$
(\tilde{\mathcal{O}}^{\Phi_a}_{k_b+k_c}) = \hat{\mathcal{O}}^{\Phi_a}_{k_b+k_c} + \frac{1}{\sqrt{N}} b^{abc}_{k_bk_c}[\hat{\mathcal{O}}^{\Phi_b}_{k_b}\hat{\mathcal{O}}^{\Phi_c}_{k_c}] + \cdots, \qquad (5.9)
$$

where the ellipses denote three particle and higher mixings. Then by construction

$$
\begin{aligned}
\langle (\tilde{\mathcal{O}}^{\Phi_a}_{k_a})^\dagger (\tilde{\mathcal{O}}^{\Phi_b}_{k_b})\rangle &= \delta^{\Phi_a\Phi_b}\delta_{k_ak_b} + \mathcal{O}\left(\frac{1}{N}\right); \qquad (5.10)\\
\langle (\tilde{\mathcal{O}}^{\Phi_a}_{k_b+k_c})^\dagger (\tilde{\mathcal{O}}^{\Phi_b}_{k_b})(\tilde{\mathcal{O}}^{\Phi_c}_{k_c})\rangle &= \frac{1}{\sqrt{N}}(C^{abc}_{k_bk_c} + b^{abc}_{k_bk_c}) + \mathcal{O}\left(\frac{1}{N}\right) \equiv \frac{1}{\sqrt{N}}\tilde{C}^{abc}_{k_bk_c} + \mathcal{O}\left(\frac{1}{N}\right).
\end{aligned}
$$

Thus to leading order in $N$ the mixed operators have the same two point functions as the single particle operators, and their three point functions still scale as $1/\sqrt{N}$. However, the structure constants are modified: $C^{abc}_{k_bk_c} \rightarrow \tilde{C}^{abc}_{k_bk_c}$. Note that the $N$ scaling of the $m$-particle term in the mixing (5.9) is $1/N^{m/2}$ such that the $n$-point functions of the mixed operators scale as $N^{(1-n/2)}$.

In the case at hand, for the exceptional extremal holographic and string correlators to agree, one needs the following quadratic operator mixings:

$$
\begin{aligned}
\mathbb{O}^{\Sigma}_2 &= \hat{\mathcal{O}}^{\Sigma}_2 + \frac{1}{2\sqrt{3N}} L(J_i, m_i)L(\bar{J}_i, \bar{m}_i)\hat{\mathcal{O}}^S_1\hat{\mathcal{O}}^S_1 + \cdots; \qquad (5.11)\\
\mathbb{O}^{S}_2 &= \hat{\mathcal{O}}^{S}_2 + \frac{1}{\sqrt{N}} L(J_i, m_i)L(\bar{J}_i, \bar{m}_i)\hat{\mathcal{O}}^S_1\hat{\mathcal{O}}^S_1 + \cdots;\\
\mathbb{O}^{S}_{\Delta+1} &= \hat{\mathcal{O}}^{S}_{\Delta+1} + \frac{\sqrt{\Delta(\Delta+1)}}{\sqrt{2N}} L(J_i, m_i)L(\bar{J}_i, \bar{m}_i)\hat{\mathcal{O}}^S_1\hat{\mathcal{O}}^S_\Delta + \cdots;\\
\mathbb{O}^{S^{(r)}}_{\Delta+1} &= \hat{\mathcal{O}}^{S^{(r)}}_{\Delta+1} + \frac{\sqrt{\Delta(\Delta+1)}}{\sqrt{2N}} L(J_i, m_i)L(\bar{J}_i, \bar{m}_i)\hat{\mathcal{O}}^S_1\hat{\mathcal{O}}^{S^{(r)}}_\Delta + \cdots;\\
\mathbb{O}^{\Sigma}_{\Delta+1} &= \hat{\mathcal{O}}^{\Sigma}_{\Delta+1} + \frac{1}{\sqrt{2N}}\frac{\Delta(\Delta+1)}{\sqrt{(\Delta-1)(\Delta+2)}} L(J_i, m_i)L(\bar{J}_i, \bar{m}_i)\hat{\mathcal{O}}^S_1\hat{\mathcal{O}}^{\Sigma}_\Delta + \cdots,
\end{aligned}
$$

where in the latter three cases $\Delta \geq 2$. The ellipses denote additional potential mixings, which include both two particle operators involving vector chiral primaries and $n$-particle operators with $n \geq 3$.

Several computations could in principle be used to verify the consistency of these operator identifications. Firstly, one could compute finite $N$ corrections to the supergravity and string theory/orbifold CFT two and three point functions, although on the supergravity side this is currently intractable since only a subset of the requisite corrections to the effective action are known. Secondly, one could compute correlation functions for operators in the same supermultiplets, which are dual to other supergravity fields. These should also be protected, and the operator identifications required for supergravity and string theory/orbifold CFT correlations functions to agree should descend from those given in (5.11).

As previously mentioned, it would be interesting to understand the non-renormalization better, both from the perspective of the 2d $\mathcal{N} = 4$ CFT and from supergravity. This could lead to other non-renormalization theorems and give insights into the required operator matching. More generally one would like to explore further the relationship between the supergravity and string theory computations, to understand better the latter. In supergravity there is by now a deep understanding of the holographic renormalization used to remove infinite volume divergences and obtain renormalized correlators. The same volume renormalization is also responsible for the finiteness of correlators in the string computations, but renormalization has not been systematically developed and applied in this context. Moreover, in supergravity there is a natural geometric understanding of the connection between boundary conditions for bulk fields and dual operators, whilst in the string computations the relation proposed in [27] between worldsheet vertex operators and CFT operators is less well understood. Thus insights from the supergravity holographic computations may help to understand further the (successful) hypotheses used in the string computations.

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